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AN ITERATION ALGORITHM FOR THE SOLUTION OF THE INVERSE BOUNDARY-VALUE PROBLEM OF HEAT CONDUCTION

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An iterative procedure is constructed for solving the inverse boundary-value problem of heat conductivity in an extremal formulation on the basis of solving a Cauchy problem.

Following [1, 2], the solution of the nonstationary heat-conduction problem

$$\bar{C}_v(T)\frac{\partial T}{\partial \tau} = \bar{\lambda}(T)\frac{\partial^2 T}{\partial X^2} + \bar{\lambda}'(T)\left(\frac{\partial T}{\partial X}\right)^2, \tag{1}$$

$$T|_{\tau=0} = 0, \tag{2}$$

$$\left.\frac{\partial T}{\partial X}\right|_{X=0} = 0,\ T|_{X=1} = T_w(\tau), \tag{3}$$

where $\tau=a_0t/R^2$, $X=x/R$, R are the dimensionless time, coordinate, and characteristic linear dimension, respectively, and $\bar{C}_v=C_v/C_{v,0}$, $\bar{\lambda}=\lambda/\lambda_0$ are the relative values of the bulk specific heat and the heat-conduction coefficient, is written in the form

$$T(X,\ \tau) = \lim_{N\to\infty}\left[\sum_{n=0}^{N-1}\Omega_n(X,\ Y_1)\,Y_{n+1}(\tau) + W(X,\ \mathbf{Y})\right],$$
$$\Omega_n(X,\ Y_1) = X^{2n}\left[\frac{\bar{C}_v(Y_1)}{\bar{\lambda}(Y_1)}\right]^n, \tag{4}$$

where the vector function $\mathbf{Y}=\{Y_1,\ Y_2,\ \ldots,\ Y_N\}$ is defined as the solution of the Cauchy problem

$$\left.\begin{aligned}
&\frac{dY_n}{d\tau} = \varepsilon_n Y_{n+1},\ n=1,\ 2,\ \ldots,\ N-1,\\
&\frac{dY_N}{d\tau} = \frac{\varepsilon_N}{\Omega_N(X, Y_1)}\left[T_w(\tau) - W(1,\ \mathbf{Y}) - \sum_{n=0}^{N-1}\Omega_n(1,\ Y_1)\,Y_{n+1}(\tau)\right],\\
&Y_n|_{\tau=0} = 0.
\end{aligned}\right\} \tag{5}$$

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Here $\varepsilon_n = 2n(2n-1)$; the function W(X, Y) takes account of the temperature dependence of the thermophysical characteristics of the material.

We assume that the thermophysical characteristics of the material depend linearly on the temperature

$$\bar{\lambda}(T) = \mu_0 + \mu_1 T,$$
$$\bar{C}_v(T) = \beta_0 + \beta_1 T, \tag{6}$$

then the function W(X, Y) in (4) takes the form

$$W(X,\ Y) = \sum_{n=0}^{\infty} X^{2n} \Phi_n(\tau), \tag{7}$$

where $\Phi_n(\tau)$ is determined from the recursion relations

$$\Phi_n(\tau) = \tilde{A}_{n-1}(\tau) + \frac{\bar{C}_v(Y_1)}{\bar{\lambda}(Y_1)} \Phi'_{n-1}(\tau) + \sum_{j=2}^{n-2} \left[\frac{\bar{C}_v(Y_1)}{\bar{\lambda}(Y_1)}\right]^{j} \tilde{A}^{(j)}_{n-1-j}(\tau),$$

$$\tilde{A}_n(\tau) = \frac{1}{\bar{\lambda}(Y_1)} \left\{\sum_{j=1}^{n} [\bar{C}_v(Y_1)]^{2j} A'_{n-j}(\tau) - \sum_{j=1}^{n} [\bar{\lambda}(Y_1)]^{2j} A_{n-j+1}(\tau) - \sum_{j=0}^{n} [\bar{\lambda}'(Y_1)]^{2j} \sum_{\nu=0}^{n-j+1} C^{2\nu+1}_{2n-2j} A_{\nu+1}(\tau) A_{n-j-\nu}(\tau)\right\},$$

$$A_0(\tau) = f(\tau),\ A_{n+1}(\tau) = [\bar{C}_v(Y_1)/\bar{\lambda}(Y_1)]\, A'_n(\tau) + \tilde{A}_n(\tau).$$

It is shown in [3] that we can set N = 3-5 in the solution (4).

In solving the inverse boundary-value heat-conduction problem (IHCP) we consider $T_w(\tau)$ the desired quantity. For a given $T_w(\tau) = u(\tau)$ we find the change in the temperature $f(\tau) = Y_1(\tau)$. Therefore, system (5) can be considered as being controlled with a control vector $U(\tau)$ [4].

One of the possible methods to obtain stable approximations of IHCP in such a formulation is the narrowing down of the domain of allowable solutions [5], i.e., this problem must be considered in a conditionally correct formulation [6]. In such an approach the IHCP is posed as an extremal problem and consists of minimizing the deviation of the computed temperatures from those given in the isolated class of desired functions. The degree of closeness between the computed and given temperatures can be judged by criteria corresponding to distances in different functional spaces.

Let us examine one possible modification. Following [7], for instance, we take the closure

$$\eta = [Y_1(\tau) - f_e(\tau)] \to \min \tag{8}$$

as a measure of the deviation of the computed value $Y_1(\tau)$ from the experimental.

Let us illustrate the algorithm presented above in an example of solving a boundary-value IHCP in such a formulation for a flat body with constant thermophysical characteristics ($W(X,\ Y) = 0$, $\Omega_n(X,\ Y_1) = X^{2n}$). We assume that the desired causal characteristic has a regular nature, i.e., is a sufficiently smooth function.

In the general case, the change of the causal characteristic in time can be arbitrarily complex. Let us consider the class of polynomial functions [7]

$$u(\tau) = u(\tau_j) + a_0^j(\tau-\tau_j) + a_1^j(\tau-\tau_j)\left(1 - \frac{\tau-\tau_j}{\Delta\tau}\right) + a_2^j(\tau-\tau_j) \times$$
$$\times \left(1 - \frac{\tau-\tau_j}{\Delta\tau}\right)\left(1 - \frac{\tau-\tau_j}{\tau^{(1)}-\tau_j}\right) + a_3^j(\tau-\tau_j)\left(1 - \frac{\tau-\tau_j}{\Delta\tau}\right)\left(1 - \frac{\tau-\tau_j}{\tau^{(1)}-\tau_j}\right)\left(1 - \frac{\tau-\tau_j}{\tau^{(2)}-\tau_j}\right) + \cdots \tag{9}$$

as a compact set M, where a_k^j are unknown coefficients.

For a symmetric heat supply, the temperature field of the body has the form (4), where the function $Y_n(\tau)$ will be defined as the integral of the system of ordinary differential equations (5). According to [7] the function $u(\tau)$ is found by iterating over steps along the τ axis. The step $\Delta\tau$ within whose limits the function $u(\tau)$ is determined should be selected from the condition that the thermal perturbations occurring in the period $\Delta\tau$ at the boundary $X = 1$ would succeed in appearing clearly at the boundary $X = 0$ at the same period of time.

At the initial time the plate was considered uniformly heated, $\Delta\tau$ was taken equal to 0.4, $\eta = 10^{-9}$, and the parameter $N = 5$. Computations were performed for different points of the section $X_* = 0,\ 0.3,\ 0.8$, and $u(\tau)$ was simulated by a second-degree polynomial. Analysis of the results shows that the surface temperature and heat flux $\left(g(\tau)=\frac{R}{\lambda}\frac{\partial T}{\partial X}\Big|_{X=1}\right)$ obtained for slowly developing thermal processes do not differ, in practice, from the exact solution (the error does not exceed 1%). The progress of the point X_* deep within the section also does not exert any substantial influence on the final result of solving the IHCP. To investigate the influence of errors on the solution of the IHCP, the input function was perturbed according to a uniform distribution law for random variables. The magnitude of the induced error was assumed to be 3, 5, and 10%. It follows from Fig. 1 that the algorithm developed can be called self-regularizing since the error in the deviation of the solution obtained from the exact value is in the corridor of the induced relative error.

The methodical computations performed for rapidly developing thermal processes also display good agreement between the results obtained and the exact solution (Fig. 2). However, progress of the point X_* deep into the section degrades the final result of the solution of the IHCP somewhat (Fig. 2). An increase in the error at the end of the interval can be explained by the sufficiently low degree of the polynomial simulating the $u(\tau)$.

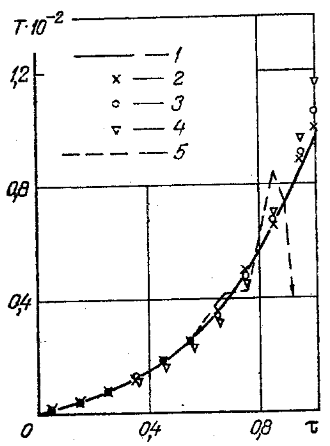

Fig. 1

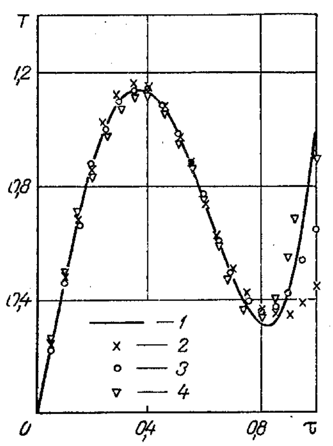

Fig. 2

Fig. 1. Restoration of the surface temperature of a flat body by interaction methods for perturbed input data ($N = 5$, $X_* = 0$): 1) exact solution [8]; 2) $\varepsilon = 3\%$; $\Delta\tau = 0.4$; 3) 5% and 0.4; 4) 10% and 0.4; 5) 0 and 0.2.

Fig. 2. Temperature change on the outer surface of a flat body, restored by iteration methods for a heat-stressed process ($N = 5$, $\Delta\tau = 0.4$): 1) exact solution [8]; 2) $X_* = 0$; 3) 0.3; 4) 0.8.

It should also be noted that the constraints are necessary for $\Delta\tau$ since an instability is observed in the solution obtained for $\Delta\tau < \Delta\tau_{per}$ (Fig. 1), i.e., the step $\Delta\tau$, within whose limits $u(\tau)$ is determined, should be in the domain of allowable values. (For the case considered above $\Delta\tau > 0.2$.) This imposes a definite constraint on the application of this method in engineering.

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